Optimization of the Modified T Vacation Policy for a Discrete-Time $$\mathrm {Geom}^{[X]}/\mathrm {G}/1$$ Queueing System with Startup

Conference Paper *in* Advances in Intelligent Systems and Computing · September 2018

DOI: 10.1007/978-3-319-66514-6_41

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Advances in Intelligent Systems and Computing 646

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Fuzzy Information and Engineering and Decision

Springer

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Optimization of the Modified T Vacation Policy for a Discrete-Time Geom$^{[X]}$/G/1 Queueing System with Startup

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Abstract. In this paper, we discuss a discrete-time Geom$^{[X]}$/G/1 queueing system with modified T vacation policy and startup time. We derive the generating functions and the mean values for the steady state system size and the waiting time, and also get those of the busy period, the vacation period and the vacation cycle by using embedded Markov chain. Finally, we determine the optimal (T^*, J^*) to minimize the cost function with fixed cost elements by constructing a cost function.

Keywords: Queueing system model · Startup time · Stochastic decomposition · Modified T vacations policy · Embedded markov chain method

1 Introduction

The server leaves for a vacation with fixed length T slots when the system is empty. After a vacation, the server returns to the system. The server immediately begins to serve if there is at least one customer waiting for service in the system; otherwise, the server takes another vacation and so on until at least one customer waits for service. This vacation policy is called T vacation policy and was firstly studied by Levy and Yechiali [1] and Heyman [2]. Sen and Gupta [3] analyzed a time dependent M/M/1 queueing with T policy via a lattice path combinatoric technique. In recent years, some authors began to study the modified T policy queueing systems. Ke [4] considered modified T vacation policy M/G/1 with an unreliable server and startup, and obtained the expected number of customers, the expected waiting time and other performances. It followed that Ke [5] studied a batch arrival queueing system under modified T vacation

B.-Y. Cao (ed.), *Fuzzy Information and Engineering and Decision*,
Advances in Intelligent Systems and Computing 646, DOI 10.1007/978-3-319-66514-6_41

policy with startup and closedown, and determined the optimal (T^*, J^*) by constructing a cost function. In addition, there are many other queueing models concerned T policy which have been studied in recent years, details of which may be seen [6–15].

In this paper, we consider a discrete time batch arrival queueing with modified T policy and startup, and derive the generating functions and the mean values for the steady state system size and the waiting time, and also get the generating functions and the expected values of the busy period, the vacation period and the vacation cycle. In addition, by constructing an cost function, we determine the optimal (T^*, J^*) to minimize the cost function. In fact, the modified T vacation policy is applied to many fields now. Take manufacturing systems for example, a machine will process a subproduct with fixed T slots after all ordinary products have been processed. And after finishing a processed subproduct while no ordinary products wait in queue, the machine continues to process another subproduct. This pattern continues cycle until at least one new ordinary product waits in the queue, otherwise it the server has already processed J subproducts. After that the machine stops to wait for arrival of the new ordinary products.

The remainder of this paper is organized as follows. A full description of the model and an embedded Markov chain are given in the Sect. 2. In Sect. 3, we obtain stochastic decomposition of the queue size and the expected values of waiting time. In Sect. 4, the expected values length of the vacation cycle, the vacation period and the busy period are obtained. We construct a cost function to introduce the optimal policy in Sect. 5. Finally in Sect. 6, we present some numerical results to illustrate the effect of λ on the expected queue size and the waiting time in the system, and obtain the optimal (T^*, J^*) with fixed cost elements.

2 Describing Model and Embedded Markov Chain

In the classical Geom$^{[X]}$/G/1 queueing system, we introduce the following vacation strategy: as soon as the system is empty, the server deactivates to take a vacation with fixed length of T. If no customers are found in the system when a vacation is finished, while the server takes another vacation with the same length T. This pattern continues cycle until a vacation is finished, the server finds at least one customer waiting in the queue or he will be already taken J vacations. If no customers are found at the end of the J-th vacation, the server stops in the system to wait for the arrival of one customer. If there is at least one customer waiting for service in the system when a vacation is finished or the server is idle in the system, he is immediately reactivated. But, the server will be need a startup time before supplying service for the waiting customers. As soon as the startup is finished, the server starts supplying service for the waiting customers until the system becomes empty again.

In the Geom$^{[X]}$/G/1 queueing model with T policy and startup time, we denote by Λ the number of customers who arrive in a single slot. The Λ is assumed to be an integral multiple, and its probability distribution and probability generating function of Λ are given by, respectively, $\lambda(k) = p(\Lambda = k), k = 0, 1, 2, \cdots; \Lambda(z) = \sum_{k=0}^{\infty} \lambda(k) z^k, |z| \leq 1$.

In addition, we denote by λ and $\lambda^{(i)}$ the mean and the i-th factorial moment of Λ, respectively, $\lambda = E[\Lambda], \lambda^{(i)} = E[\Lambda(\Lambda-1)\cdots(\Lambda-i+1)], i = 2, 3, \cdots$.

Let X be the service time of one customer and the length of the service time be an integral multiple of a slot duration, then its probability distribution and probability generating function are given by, respectively, $b(l) = p(X = l),\ l = 1, 2, \cdots; B(z) = \sum_{l=1}^{\infty} b(l) z^l, |z| \leq 1$.

Let b and $b^{(i)}$ be the mean and the i-th moment of the service time distribution, respectively, $b = E[X]; b^{(l)} = E[X^l],\ i = 2, 3, \cdots$.

Let S be the startup time and the length of the startup time be an integral multiple of a slot duration, then its probability distribution and probability generating are given by, respectively, $s(l) = p(S = l), l = 1, 2, \cdots; S(z) = \sum_{l=1}^{\infty} s(l) z^l, |z| \leq 1$.

Let s and $s^{(i)}$ be the mean and the ith factorial moment of the startup time distribution, respectively: $s = E[S]; s^{(i)} = E[S(S-1)\cdots(S-i+1)], i = 2, 3, \cdots$.

Now we consider a Markov chain $\{L_n; n = 1, 2, \cdots\}$, where L_n denotes the number of customers present in the system after the server has completed service for the n-th customer. And suppose that A_n is the number of arriving customers during the n-th customer's service and α is that of present customers in the system at the end of the startup time, thus we have

$$L_{n+1} = \begin{cases} L_n + A_{n+1} - 1, & L_n \geq 1, \\ \alpha + A_{n+1} - 1, & L_n = 0 \end{cases}$$

Let $A(z)$ be the PGF for A_n, and $\alpha(z)$ for α. For the system, we imagine a Geom$^{[X]}$/G/1 queueing system with a vacation period that may terminate in one of the following two situations.

Case 1. If there is at least one customer waiting in the system at the end of the j-th vacation ($1 \leq j \leq J$), the server immediately operates a startup. In this case, at the end of the startup time the PGF for the number of customers waiting in the system is given by $[1-\lambda^{JT}(0)][\Lambda^T(z)-\lambda^T(0)][1-\lambda^T(0)]^{-1} S[\Lambda(z)]$.

Case 2. If there is no customer waiting in the system at the end of the J-th vacation, the server stays idle in the system. Once a customer arrives, the server immediately operates a startup. Thus, in this case, at the end of the startup time the PGF for the number of customers found of in the system is given by $\lambda^{JT}(0)[\Lambda(z) - \lambda(0)][1 - \lambda(0)]^{-1} S[\Lambda(z)]$.

From the two cases above, at the end of the startup time the PGF $\alpha(z)$ for the number of customers waiting in the system is given by

$$\alpha(z) = [1 - \lambda^{JT}(0)] \frac{\Lambda^T(z) - \lambda^T(0)}{1 - \lambda^T(0)} S[\Lambda(z)] + \lambda^{JT}(0) \frac{\Lambda(z) - \lambda(0)}{1 - \lambda(0)} S[\Lambda(z)] \quad (1)$$

If we denote by $\{k_j, j = 0, 1, 2, \cdots\}$ and $\{b_j, j = 0, 1, 2, \cdots\}$ the probability distributions for A_n and $\alpha + A_n - 1$, respectively, then the PGFs for them are given by $A(z) = \sum_{j=0}^{\infty} k_j z_j = B[\Lambda(z)], \xi(z) = \sum_{j=0}^{\infty} b_j z_j = \frac{\alpha(z) B[\Lambda(z)]}{z}$.

Therefore, the transition probability matrix of Markov chain $\{L_n, n = 1, 2, \cdots\}$ is given by

$$\tilde{P} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ k_0 & k_1 & k_2 & k_3 & \cdots \\ & k_0 & k_1 & k_2 & \cdots \\ & & k_0 & k_1 & \cdots \\ & & & \vdots & \vdots \end{bmatrix}.$$

By the Foster rule, we can prove that the Markov chain $\{L_n, n = 1, 2, \cdots\}$ is positive recurrence if and only if $\rho = \lambda b < 1$.

3 Stochastic Decomposition of Queue Size and Expected Waiting Times in System

In the section, we will obtain the PGFs for the steady-state system size and the waiting time.

Theorem 1. *If $\rho < 1$, the steady-state system size L can be decomposed into the sum of two stochastic independent variables, i.e., $L = L_{Geom^{[X]}/G/1} + L_d$, where $L_{Geom^{[X]}/G/1}$ denotes the steady-state system size of classical $Geom^{[X]}/G/1$ model which generating function and expected value have been given in [15]. Then*

$$L_d(z) = \frac{\lambda[1 - \alpha(z)]}{E[\alpha][1 - \Lambda(z)]}$$

is the generating function of additional system L_d.

Proof. We assume that a steady-state distribution exists for the Markov chain $\{L_n; n = 1, 2, \cdots\}$ and that it is denoted by $\pi_k = \lim_{n\to\infty} p(L_n = k), k = 0, 1, 2, \cdots$.

Because the steady-state $\{\pi_k, k \geq 0\}$ satisfies $\Pi \widetilde{P} = \Pi$, we have

$$\pi_j = \pi_0 b_j + \sum_{i=1}^{j+1} \pi_i k_{j+1-i},\ j \geq 0$$

where $\Pi = (\pi_0, \pi_1, \pi_2, \cdots)$.

Taking generating function, we obtain

$$\begin{aligned} L(z) &= \textstyle\sum_{j=0}^{\infty} \pi_j z^j = \pi_0 \sum_{j=0}^{\infty} b_j z^j + \sum_{j=0}^{\infty} \sum_{i=1}^{j+1} \pi_i k_{j+1-i} z^j \\ &= \pi_0 \tfrac{\alpha(z) B[\Lambda(z)]}{z} + \tfrac{1}{z} B[\Lambda(z)][L(z) - \pi_0] \end{aligned} \tag{2}$$

Substituting Eq.(1) into Eq.(2), we get

$$L(z) = \frac{\pi_0 B[\Lambda(z)][\alpha(z) - 1]}{z - B[\Lambda(z)]}$$

By the normalization $L(1) = 1$ and the L'Hospital rule, we obtain

$$\pi_0 = \frac{1 - \rho}{E[\alpha]}$$

where

$$E[\alpha] = [1 - \lambda^{JT}(0)]\frac{T\lambda + s\lambda[1 - \lambda^T(0)]}{1 - \lambda^T(0)} + \lambda^{JT}(0)\frac{\lambda + s\lambda[1 - \lambda(0)]}{1 - \lambda(0)}$$

is the mean number customers at the end of the startup time.

Substituting π_0 into Eq.(2), we obtain

$$L(z) = \frac{(1-\rho)B[\Lambda(z)][\alpha(z) - 1]}{E[\alpha]\{z - B[\Lambda(z)]\}} = L_{Geom^{[X]}/G/1} \cdot \frac{\lambda[1 - \alpha(z)]}{E[\alpha][1 - \Lambda(z)]} \tag{3}$$

Thus, it yields

$$L_d(z) = \frac{\lambda[1 - \alpha(z)]}{E[\alpha][1 - \Lambda(z)]}$$

The proof is complete.

In addition, from the Theorem 1, we obtain the mean queue size in system given by

$$E[L] = E[L_{Geom^{[X]}/G/1}] + E[L_d] = E[L_{Geom^{[X]}/G/1}] + \frac{2\lambda E[\alpha(\alpha - 1)] - \lambda^{(2)}E[\alpha]}{2\lambda E[\alpha]}$$

where

$$\begin{aligned} E[\alpha(\alpha - 1)] = {} & \tfrac{1-\lambda^{JT}(0)}{1-\lambda^T(0)}\{T(T-1)\lambda^2 + T\lambda^{(2)} + 2Ts\lambda^2 + [1 - \lambda^T(0)](\lambda^2 s^{(2)} + \lambda^{(2)}s)\} \\ & + \tfrac{\lambda^{JT}(0)}{1-\lambda(0)}\{2\lambda^2 s + \lambda^{(2)} + [1 - \lambda(0)](\lambda^2 s^{(2)} + \lambda^{(2)}s)\} \end{aligned}$$

Theorem 2. *If $\rho < 1$, the steady-state waiting time W can be decomposed into the sum of two stochastic independent variables, i.e., $W = W_{Geom^{[X]}/G/1} + W_d$, where $W_{Geom^{[X]}/G/1}$ denotes the steady-state waiting time of classical Geom$^{[X]}$/G/1 model which generating function and expected value have been given in [15]. Then*

$$W_d(z) = \frac{[1 - \lambda(0)][1 - \beta(z)]}{E(\alpha)(1 - z)}$$

is the generating function of additional system W_d.

Proof. We consider the waiting time of an arbitrary customer in FCFS systems and give explicit expressions for the PGF $W(z)$ of the waiting time for FCFS systems. The distribution of the waiting time can be easily obtained by assuming that a group of customers arrive in the same slot and they constitute one super-customer in a Geom/G/1 system. That is, the PGF $\Lambda_g(z)$ and the mean λ_g for the number of the super-customers who arrive in a slot in the Geom/G/1 system are given by, respectively,

$$\Lambda_g(z) = \lambda(0) + [1 - \lambda(0)]z \tag{4}$$

$$\lambda_g = 1 - \lambda(0) \tag{5}$$

The PGF $B_g(z)$ for the service time of a super-customer is given by

$$B_g(z) = \widetilde{\Lambda}[B(z)] = \frac{\Lambda[B(z)] - \lambda(0)}{1 - \lambda(0)} \tag{6}$$

Therefore, the PGF for the number of present super-customer in the corresponding $\text{Geo}^{[X]}/\text{G}/1$ system at the end of super-customer's service is given by

$$L_g(z) = \frac{(1-\rho)B_g[\Lambda_g(z)][\alpha_g(z) - 1]}{E[\alpha]\{z - B_g[\Lambda_g(z)]\}} \tag{7}$$

where

$$\alpha_g(z) = [1 - \lambda^{JT}(0)]\frac{\Lambda_g^T(z) - \lambda^T(0)}{1 - \lambda^T(0)}S[\Lambda_g(z)] + \lambda^{JT}(0)\frac{\Lambda_g(z) - \lambda(0)}{1 - \lambda(0)}S[\Lambda_g(z)]$$

Let $\beta[\Lambda_g(z)] = \alpha_g(z)$, then we obtain

$$\beta(z) = [1 - \lambda^{JT}(0)]\frac{z^T - \lambda^T(0)}{1 - \lambda^T(0)}S(z) + \lambda^{JT}(0)\frac{z - \lambda(0)}{1 - \lambda(0)}S(z)$$

Since the number of present super-customer in the system at the end of super-customer's service equals just that arriving super-customer in the time interval that they have been in the system, by $W_g(z)$ denoting the PGF for the waiting time of the super-customer, we have the following expression

$$L_g(z) = W_g[\Lambda_g(z)]B[\Lambda_g(z)] \tag{8}$$

Note that the traffic intensity ρ is the same in classic Geom/G/1 and $\text{Geom}^{[X]}/\text{G}/1$ queue systems. Substituting Eqs.(4)–(7) into Eq.(8), we get the PGF $W_g(z)$ for the waiting time W_g of a supercustomer in an FCFS system as

$$W_g(z) = \frac{(1-\rho)[1-\lambda(0)][1-\beta(z)]}{E[\alpha]\{\Lambda[B(z)] - z\}} \tag{9}$$

The waiting time W of an arbitrary customer consists of two independent components. One is the waiting time W_g of a super-customer to who the arbitrary customer belongs; the other, denoted by J, is the sum of the service time for those customers within the same super-customer who are served in front of the arbitrary customer. Note that these components are independent. If $J(z)$ denotes the PGF for J, we have

$$W(z) = W_g(z)J(z) \tag{10}$$

In order to get the $J(u)$, we know that the number of customers within the super-customer that are served in front of the arbitrary customer is equivalent to the forward recurrence time in a discrete-time renewal process when the interrenewal time is given by the number of customers included in the super-customer. Hence we have

$$J(z) = \frac{1 - \Lambda[B(z)]}{\lambda[1 - B(z)]} \tag{11}$$

Substituting Eqs. (9) and (11) into Eq.(10), we get

$$W(z) = \frac{(1-\rho)[1-\lambda(0)][1-\beta(z)]\{1-\Lambda[B(z)]\}}{\lambda E[\alpha][1-B(z)]\{\Lambda[B(z)]-z\}} = W_{Geo^{[x]}/G/1}(z)W_d(z)$$

Thus, it yields

$$W_d(z) = \frac{[1-\lambda(0)][1-\beta(z)]}{E(\alpha)(1-z)}$$

The proof is complete.

In addition, from the Theorem 2, we obtain the mean waiting time in system given by

$$E[W] = E[W_{Geo^{[x]}/G/1}] + E[W_d] = E[W_{Geo^{[x]}/G/1}] + \frac{E[\beta(\beta-1)][1-\lambda(0)]}{2E(\alpha)}$$

where

$$E[\beta(\beta-1)] = \frac{1-\lambda^{JT}(0)}{1-\lambda^T(0)}\{T(T-1)+2Ts+[1-\lambda^T(0)]s^{(2)}\}+\frac{\lambda^{JT(0)}}{1-\lambda(0)}\{2s+[1-\lambda(0)]s^{(2)}\}$$

4 Expected Length of the Vacation Cycle, the Vacation Period and the Busy Period

We define a time interval as a vacation period that starts at the busy period and terminates at the beginning of the startup time, and denote it by I_v. It consists of a vacation and an idle period. Then we can obtain the probabilities and the PGF, respectively,

$$\begin{cases} P(I_v = kT) = [1-\lambda(0)]\lambda^{(k-1)T}(0)\sum_{j=0}^{T-1}\lambda^j(0), \ 1 \le k \le J, \\ P(I_v = JT+i) = \lambda^{JT+i-1}(0)[1-\lambda(0)], \ i \ge 1 \end{cases}$$

and

$$\begin{aligned} I_v(z) &= \textstyle\sum_{j=1}^{\infty} P(I_v = j)z^j \\ &= [1-\lambda(0)]\{\textstyle\sum_{k=1}^{J}\lambda^{(k-1)T}(0)\sum_{j=0}^{T-1}\lambda^j(0)z^{kT} + \sum_{i=1}^{\infty}\lambda^{JT+i-1}(0)z^{JT+i}\} \\ &= \tfrac{[1-\lambda(0)][1-\lambda^{JT}(0)z^{JT}]z^T}{1-\lambda^T(0)z^T} + \tfrac{\lambda^{JT}(0)[1-\lambda(0)]z^{JT+1}}{1-\lambda(0)z} \end{aligned}$$

Thus, it leads to the mean vacation period length

$$E(I_v) = \frac{-JT\lambda^{JT}(0)[1-\lambda(0)] + T[1-\lambda^{JT}(0)]}{1-\lambda^T(0)} + \frac{\lambda^{JT}(0)\{(JT+1)[1-\lambda(0)]+\lambda(0)\}}{1-\lambda(0)} \quad (12)$$

We denote by Θ_v a busy period defined as a time interval from the end of the startup time to the beginning of the next vacation. Since α is the number of customers in the system at the beginning of a busy period, the PGF $\Theta_v(z)$ and the mean $E(\Theta_v)$ for the length Θ_v of a busy period are given by

$$\begin{aligned}\Theta_v(z) &= \alpha[\Theta(z)]\\ &= [1-\lambda^{JT}(0)]\tfrac{\Lambda^T(\Theta(z))-\lambda^T(0)}{1-\lambda^T(0)}S[\Lambda(\Theta(z))]+\lambda^{JT}(0)\tfrac{\Lambda(\Theta(z))-\lambda(0)}{1-\lambda(0)}S[\Lambda(\Theta(z))]\end{aligned}$$

and

$$\begin{aligned}E(\Theta_v) &= \tfrac{E(\Theta)[1-\lambda^{JT}(0)]}{1-\lambda^T(0)}\{\lambda T+s\lambda[1-\lambda^T(0)]\}+\tfrac{\lambda E(\Theta)\lambda^{JT}(0)\{1+s[1-\lambda(0)]\}}{1-\lambda(0)}\\ &= \tfrac{\rho[1-\lambda^{JT}(0)]}{(1-\rho)[1-\lambda^T(0)]}\{T+s[1-\lambda^T(0)]\}+\tfrac{\rho\lambda^{JT}(0)\{1+s[1-\lambda(0)]\}}{(1-\rho)[1-\lambda(0)]}\end{aligned} \tag{13}$$

where Θ is the length of a busy period caused by the service time of a single customer in the system, $\Theta(z)$ and $E(\Theta)$ are the PGF and the mean of Θ, respectively.

A vacation cycle consists of a vacation period, startup time and the follow busy period. The PGF $C_v(z)$ and the mean $E[C_v]$ for the length C_v of the vacation cycle are given by

$$\begin{aligned}C_v(z) &= I_v(z)\cdot S(z)\cdot\Theta_v(z)\\ &= S(z)\times S[\Lambda(\Theta(z))]\times\{\tfrac{[1-\lambda^T(0)][1-\lambda^{JT}(0)z^{JT}]z^T}{1-\lambda^T(0)z^T}+\tfrac{\lambda^{JT}(0)[1-\lambda(0)]z^{JT+1}}{1-\lambda(0)z}\}\\ &\times\{[1-\lambda^{JT}(0)]\tfrac{\Lambda^T(\Theta(z))-\lambda^T(0)}{1-\lambda^T(0)}+\lambda^{JT}(0)\tfrac{\Lambda(\Theta(z))-\lambda(0)}{1-\lambda(0)}\}\end{aligned}$$

and

$$\begin{aligned}E[C_v] &= E(I_v)+s+E(\Theta_v)\\ &= s+\tfrac{-JT\lambda^{JT}(0)[1-\lambda^T(0)]+T[1-\lambda^{JT}(0)]}{1-\lambda^T(0)}+\tfrac{\lambda^{JT}(0)\{(JT+1)[1-\lambda(0)]+\lambda(0)\}}{1-\lambda(0)}\\ &+\tfrac{\rho[1-\lambda^{JT}(0)]}{(1-\rho)[1-\lambda^T(0)]}\{T+s[1-\lambda^T(0)]\}+\tfrac{\rho\lambda^{JT}(0)\{1+s[1-\lambda(0)]\}}{(1-\rho)[1-\lambda(0)]}\end{aligned} \tag{14}$$

5 Optimal Policy

In this section, we will construct a total long-run average cost function per customer per unit time for the system, in which T and J are all decision variables. Our purpose is to determine the optimal T and J to minimize this cost function. The following cost elements are considered: c_h is the holding cost per unit time for each present customer in the system; c_s is the setup cost for per busy cycle; c_i is the cost per unit time for keeping the server off; c_u is the startup cost per unit time for the preparatory work of the server before starting the service.

Employing the definition of each cost element and its corresponding system characteristics, the total long-run average cost per unit time is given by

$$\begin{aligned}F(J,T) &= c_hE[L]+c_s\tfrac{1}{E[C_v]}+c_i\tfrac{E[I_v]}{E[C_v]}+c_u\tfrac{s}{E[C_v]}\\ &= c_hE[L_{Geo^{[X]}/G/1}]+\tfrac{c_h\{2\lambda E[\alpha(\alpha-1)]-\lambda^{(2)}E[\alpha]\}}{2\lambda E[\alpha]}+\tfrac{c_s(1-\rho)(1-\lambda(0))[1-\lambda^T(0)]}{A}\\ &+\tfrac{c_i(1-\rho)(1-\lambda(0))[1-\lambda^T(0)]E[I_v]}{A}+\tfrac{c_s s(1-\rho)(1-\lambda(0))[1-\lambda^T(0)]}{A}\end{aligned} \tag{15}$$

where

$$E[L_{Geo^{[x]}/G/1}] = \rho + \frac{\lambda^2 b^{(2)} - \lambda\rho + b\lambda^{(2)}}{2(1-\rho)}$$

$$\begin{aligned} A = {} & s(1-\rho)[1-\lambda(0)][1-\lambda^T(0)] + (1-\rho)[1-\lambda(0)]\{-JT\lambda^{JT}(0)[1-\lambda^{JT}(0)]\} \\ & +T[1-\lambda^{JT}(0)] + (1-\rho)\lambda^{JT}(0)[1-\lambda^T(0)]\{(JT+1)[1-\lambda(0)]+\lambda(0)\} \\ & +\rho[1-\lambda(0)][1-\lambda^{JT}(0)]\{T+s[1-\lambda^T(0)]\} + \rho\lambda^{JT}(0)[1-\lambda^{JT}(0)]\{1+s[1-\lambda(0)]\} \end{aligned}$$

We consider the model with a minimum cost function. For fixed c_s, c_h, c_i and c_u, the optimization problem is described as follows:

$$\begin{aligned} & \min \quad F(J,T) = c_h E[L] + c_s \tfrac{1}{E[C_v]} + c_i \frac{E[I_v]}{E[C_v]} + c_u \tfrac{s}{E[C_v]}, \\ & \text{s.t.} \quad T \geq 1,\ J \geq 1, T, J \in N^+, \quad \text{and} \quad (c_h, c_s\ c_i,\ c_u > 0) \end{aligned} \tag{16}$$

We denote the solution by (J^*, T^*) that minimizes the cost function $F(J,T)$.

6 Numerical Illustration

In the section, the first purpose is to study the effects of some parameters on the expected values of the customers' number and waiting time in the system. We assume that the number of customers Λ in a single slot follows a poisson distribution with a parameter λ, and that service time X of a customer and setup time S follow geometric distributions with the parameters p_1 and p_2, respectively.

For convenience, we choose $T = 1, 10, 20, J = 5, p_1 = 0.8$ and $p_2 = 0.8$, vary the value of λ from 0.3 to 0.7.

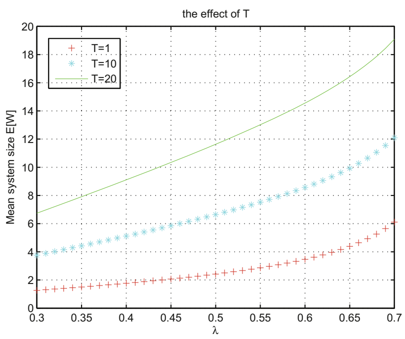

Fig. 1. The expected system size

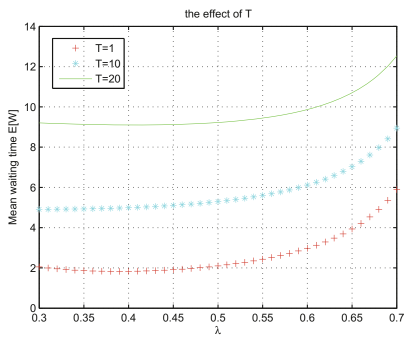

Fig. 2. The expected waiting time

Figures 1 and 2 show that the expected system size and the expected waiting time are all functions of the arrival rate λ. We find that whenever λ increases, the expected system size and the expected waiting time increase at a higher level. Meanwhile, the both increase faster with T increasing.

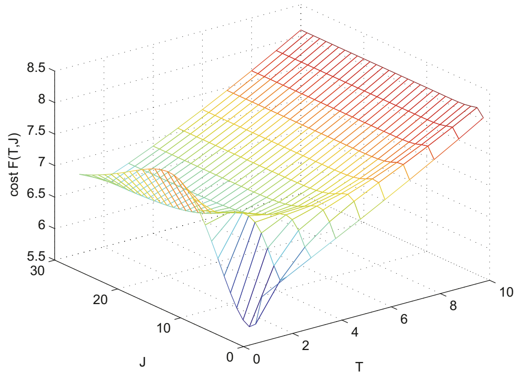

Fig. 3. Mean waiting time $E[W_v]$ versus traffic intensity ρ.

The second purpose is to study the effects of some parameters on the cost function. We assume that the number of customers Λ in a single slot follows a poisson distribution with a parameter λ, and that service time X of a customer and setup time S follow geometric distributions with parameters p_1 and p_2, respectively. We choose $\lambda = 0.15, p_1 = 0.3, p_2 = 0.5, c_h = 2, c_s = 20, c_i = 3$ and $c_u = 10$, vary the values of T and J from 1 to 10 and 1 to 30, respectively.

Figure 3 shows that the minimum cost value per unit time of 5.5731 is obtained at $(T^*, J^*) = (1, 3)$.

7 Conclusion

The paper introduces the optimal modified T vacation policy for the discrete-time Geom$^{[X]}$/G/1 queueing with startup. By using the embedded Markov chain method, we obtain the PGFs and the expected values for the steady state system size, waiting time, busy period and vacation cycle. Additionally, By constructing a cost function, we determine the optimal values of T and J to minimize the cost function. We will further try to study the N policy for the Geom$^{[X]}$/G/1 queueing system.

Acknowledgements. Thanks to the support by National Natural Science Foundation of China (No. 61175073).

Recommender: Professor Lv Shengli, School of Science, Yanshan University, Qinhuangdao 066004 P.R. China.

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